Math 2A – Chapter 13 Test – Fall 07. Name________________________________
Show your work for credit. Write all responses on separate paper. Do not abuse a calculator.

1. Evaluate $\oint_C y\,dx - x\,dy$ around the ellipse $x = 2\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$

2. Consider a tetrahedron with vertices at $P_0 = (0, 0, 0)$, $P_1 = (1, 0, 1)$, $P_2 = (1, 0, -1)$, and $P_3 = (1, 1, 0)$. Find the flux of $\vec{F} = \langle y, 0, 0 \rangle$ through
   a. the face $P_0P_1P_2$.
   b. the face $P_0P_1P_3$.

3. Let $f(x, y, z) = 1/\rho = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$
   a. Calculate $\vec{F} = \overrightarrow{\nabla f}$ and describe the gradient field geometrically.
   b. Calculate the flux of $\vec{F}$ over a sphere of radius a centered at the origin.
   c. Show that $\operatorname{div}(\vec{F}) = 0$.
   Does the result of (b) then contradict the divergence theorem (Gauss' Theorem) ? Explain.

4. Show that $\operatorname{div}\left(f\overrightarrow{\nabla f}\right) = \left|\overrightarrow{\nabla f}\right|^2 + f\nabla^2 f$

5. Evaluate the surface integral $\iint_S \sqrt{1 + x^2 + y^2}\,dS$ where S is the helicoid with vector equation
   $r(u, v) = \langle u\cos v, u\sin v, v \rangle$ and $0 \le u \le 1$, $0 \le v \le \pi$.

6. Let $\vec{F} = \langle a\sin z + bxy^2, 2x^2y, x\cos z - z^2 \rangle$
   a. Compute, in terms of the constants a and b the work done by the vector field $\vec{F}$ along the portion of the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ from (1,0,0) to (1,0,2π)
   b. Compute $\operatorname{curl}(\vec{F})$. Show that $\vec{F}$ is conservative only if $a = 1$ and $b = 2$.
   c. Find the potential function for $\vec{F}$ using $a = 1$, $b = 2$ and verify your answer to part (a) using the Fundamental Theorem of Calculus.

7. Evaluate $\iint_S x\,dS$ where S is the part of the plane $z = x$ that lies above the square with the vertices (0,0), (1,0), (0,1) and (1,1).

8. Let $\vec{F} = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle$, Evaluate $\int_C \vec{F} \cdot \overrightarrow{dr}$ where C is the elliptical path $\vec{r}(t) = \langle \cos t, \sin t, \cos t \rangle$

Math 2A – Chapter 13 Test Solutions – Fall 07

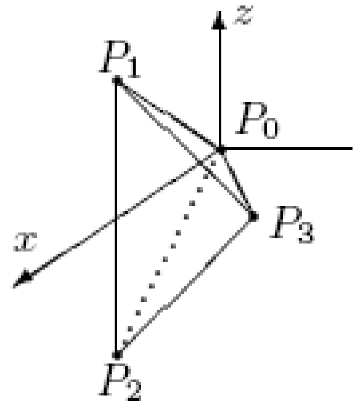
1. Evaluate $\oint_C ydx - xdy$ around the ellipse $x = 2\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$

   SOLN: $\oint_C ydx - xdy = \int_0^{2\pi} -6\sin^2 t - 6\cos^2 t dt = \int_0^{2\pi} -6dt = -12\pi$

   You can apply Green's theorem here, if you like:

   $$\oint_C ydx - xdy = \oint_C \langle y, -x\rangle \cdot \overrightarrow{dr} = \iint_D (-1-1)\, dA = -2\int_{-2}^{2}\int_{-\frac{3}{2}\sqrt{4-x^2}}^{\frac{3}{2}\sqrt{4-x^2}} dydx = -6\int_{-2}^{2}\sqrt{4-x^2}\,dx = -12\pi$$

2. Consider a tetrahedron with vertices at $P_0 = (0, 0, 0)$, $P_1 = (1, 0, 1)$, $P_2 = (1, 0, -1)$, and $P_3 = (1, 1, 0)$. Find the flux of $\vec{F} = \langle y, 0, 0\rangle$ through



   a. the face $P_0P_1P_2$.
      SOLN: This face is contained in the xz-plane and since the field vectors are parallel to this plane, there is no flux through that plane.
   b. the face $P_0P_1P_3$.
      SOLN: A normal to this face is
      $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_3} = \langle 1,0,1\rangle \times \langle 1,1,0\rangle = \langle -1,1,1\rangle$.

      Thus

      $$\iint_S \vec{F}\cdot\hat{n}dS = \iint_S \langle y,0,0\rangle \cdot \langle -1,1,1\rangle dA = \iint_S -ydA$$
      $$= \int_0^1\int_0^x -ydydx = -\int_0^1 \frac{x^2}{2}dx = -\frac{1}{6}$$

As a follow-up, it may be noted that the flux through the face $P_0P_2P_3$ is also –1/6, by symmetry. Thus, if Gauss' divergence theorem is to be believed, since the divergence of the vector field is zero, the flux through the face $P_1P_2P_3$ must be 1/3. Let's see: $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \langle 0,0,-2\rangle \times \langle 0,1,-1\rangle = \langle 2,0,0\rangle$, so the unit normal is $\langle 1,0,0\rangle$ which makes sense since this face is in the plane $x = 1$. So the flux through that face is indeed $\iint_S \vec{F}\cdot\hat{n}dS = \iint_S \langle y,0,0\rangle \cdot \langle 1,0,0\rangle dS = \iint_S ydS = \int_0^1\int_{y-1}^{1-y} ydzdy = \int_0^1 y(2-2y)\,dy = y^2 - \frac{2y^3}{3}\Big|_0^1 = \frac{1}{3}$

3. Let $f(x,y,z) = 1/\rho = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

   a. Calculate $\vec{F} = \overrightarrow{\nabla f}$ and describe the gradient field geometrically.

      SOLN: $\overrightarrow{\nabla f} = \left\langle \frac{-x}{\rho^3}, \frac{-y}{\rho^3}, \frac{-z}{\rho^3}\right\rangle$

   b. Calculate the flux of $\vec{F}$ over a sphere of radius a centered at the origin.

      $$\oiint_S \vec{F}\cdot\hat{n}dS = \oiint_S \left\langle \frac{-\sin\phi\cos\theta}{a^2}, \frac{-\sin\phi\sin\theta}{a^2}, \frac{-\cos\phi}{a^2}\right\rangle \cdot \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle dS$$

      SOLN:
      $$= \frac{-1}{a^2}\int_0^\pi\int_0^{2\pi}\left(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi\right)a^2\sin\phi d\theta d\phi$$
      $$= -\int_0^\pi d\theta \int_0^{2\pi}\sin\phi d\phi = -2\pi(2) = -4\pi$$

c. Show that $\operatorname{div}(\vec{F}) = 0$.

Does the result of (b) then contradict the divergence theorem (Gauss' Theorem) ? Explain.

SOLN:
$$\operatorname{div}(\vec{F}) = \left\langle \frac{\partial}{\partial x}\frac{-x}{\rho^3}, \frac{\partial}{\partial y}\frac{-y}{\rho^3}, \frac{\partial}{\partial z}\frac{-z}{\rho^3} \right\rangle = \frac{-\rho^3 + x3x\rho}{\rho^6} + \frac{-\rho^3 + y3y\rho}{\rho^6} + \frac{-\rho^3 + z3z\rho}{\rho^6}$$
$$= \frac{-3\rho^3 + 3(x^2+y^2+z^2)\rho}{\rho^6} = \frac{-3\rho^3 + 3\rho^2\rho}{\rho^6} = 0$$

The equation of the divergence theorem is that $\oiint_S \vec{F}\cdot\hat{n}dS = \iiint_E \vec{\nabla}\cdot\vec{F}dV$ which in this case evidently leads to the contradiction, $-4\pi = 0$. However, this does not contradict the theorem since a premise of the theorem is that the vector field have continuous partial derivatives on an open region containing *E*. This vector field is not even defined at the origin, never mind having continuous partials.

4. Show that $\operatorname{div}(f\overrightarrow{\nabla f}) = \left|\overrightarrow{\nabla f}\right|^2 + f\nabla^2 f$

SOLN:
$$\operatorname{div}(f\overrightarrow{\nabla f}) = \operatorname{div}(\langle ff_x, ff_y, ff_z\rangle) = ff_{xx} + f_xf_x + ff_{yy} + f_yf_y + ff_{zz} + f_zf_z =$$
$$= f(f_{xx} + f_{yy} + f_{zz}) + \langle f_x, f_y, f_z\rangle\cdot\langle f_x, f_y, f_z\rangle = f\nabla^2 f + \left|\overrightarrow{\nabla f}\right|^2$$

5. Evaluate the surface integral $\iint_S \sqrt{1+x^2+y^2}\,dS$ where *S* is the helicoid with vector equation $r(u,v) = \langle u\cos v, u\sin v, v\rangle$ and $0 \le u \le 1$, $0 \le v \le \pi$.

SOLN:
$$|\vec{r}_u \times \vec{r}_v| = |\langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle| = |\langle \sin v, -\cos v, u\rangle| = \sqrt{1+u^2}$$
$$\Rightarrow \iint_S \left(\sqrt{1+u^2}\right)^2 dS = \int_0^\pi \int_0^1 (1+u^2)\,du\,dv = \frac{4\pi}{3}$$

6. Let $\vec{F} = \langle a\sin z + bxy^2, 2x^2y, x\cos z - z^2\rangle$

a. Compute, in terms of the constants *a* and *b* the work done by the vector field $\vec{F}$ along the portion of the helix $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$ from (1,0,0) to (1,0,2π)

SOLN:
$$\int_C \vec{F}\cdot\overrightarrow{dr} = \int_0^{2\pi} \langle a\sin t + b\cos t\sin^2 t, 2\cos^2 t\sin t, \cos^2 t - t^2\rangle\langle -\sin t, \cos t, 1\rangle dt =$$
$$= \int_0^{2\pi} -a\sin^2 t - b\cos t\sin^3 t + 2\cos^3 t\sin t + \cos^2 t - t^2 dt$$
$$= -a\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)_0^{2\pi} - b\left(\frac{1}{4}\sin^4 t\right)_0^{2\pi} + \left(-\frac{1}{2}\cos^4 t\right)_0^{2\pi} + \left(\frac{t}{2} + \frac{\sin 2t}{4}\right)_0^{2\pi} + \left(\frac{t^3}{3}\right)_0^{2\pi} = (1-a)\pi - \frac{8\pi^3}{3}$$

b. Compute $\operatorname{curl}(\vec{F})$. Show that $\vec{F}$ is conservative only if *a* = 1 and *b* = 2.

SOLN: $\operatorname{curl}(\vec{F}) = \langle 0, a\cos z - \cos z, 4xy - 2bxy\rangle = \langle 0,0,0\rangle$ only if *a* = 1 and *b* = 2.

c. Find the potential function for $\vec{F}$ using $a = 1$, $b = 2$ and verify your answer to part (a) using the Fundamental Theorem of Calculus.

SOLN:
$$f(x,y,z) = x\sin z + x^2y^2 - \frac{1}{3}z^3 \Rightarrow \int_C \overrightarrow{\nabla f}\cdot\overrightarrow{dr} = \int_0^{\pi} \overrightarrow{\nabla f}\cdot\langle -\sin t, \cos t, 1\rangle dt$$
$$= f(x(2\pi), y(2\pi), z(2\pi)) - f(x(0), y(0), z(0)) = -\frac{8\pi^3}{3}$$

7. Evaluate $\iint_S x\,dS$ where S is the part of the plane $z = x$ that lies above the square with the vertices (0,0), (1,0), (0,1) and (1,1).

SOLN: $\iint_S x\,dS = \int_0^1\int_0^1 x\sqrt{2}\,dx\,dy = \frac{\sqrt{2}}{2}$

8. Let $\vec{F} = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3\rangle$, Evaluate $\int_C \vec{F}\cdot\overrightarrow{dr}$ where C is the elliptical path $\vec{r}(t) = \langle \cos t, \sin t, \cos t\rangle$

SOLN: This is (almost obviously) the gradient field for the potential function $f(x,y,z) = x^2y^3z^4$, thus it is a conservative vector field and the path integral around any closed path is zero. It's also possible to compute the curl, which is also the zero vector.

Oh no! It seems some of you wanted to compute this thing directly…well: I'll try:

$$\int_C \vec{F}\cdot\overrightarrow{dr} = \int_0^{2\pi} \langle 2\cos^5 t\sin^3 t, 3\cos^6 t\sin^2 t, 4\cos^5 t\sin^3 t\rangle\cdot\langle -\sin t, \cos t, -\sin t\rangle dt$$
$$= \int_0^{2\pi}\left(-6\cos^5 t\sin^4 t + 3\cos^7 t\sin t\right)dt = \int_0^{2\pi}\left(-6\left(1-\sin^2 t\right)^2\sin^4 t\cos t\right)dt + \int_0^{2\pi} 3\cos^7 t\sin t\,dt$$
$$= -6\int_0^0\left(1-u^2\right)^2 u^4\,du - 3\int_1^1 v^7\,dv = 0$$

9. Another statement of Stokes' theorem goes like this:

Let S be the graph of the function $z = f(x, y)$, where $f(x, y)$ is defined on some region S for which Green's Theorem is true. Further, let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ be a vector field on* S. *Then, under suitable differentiability and orientation assumptions, the following holds*

$$\oint_{\partial S} \vec{F}\cdot\hat{T}ds = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\,dS = \iint_D \left[\left(R_y - Q_z\right)\cos\alpha + \left(P_z - R_x\right)\cos\beta + \left(Q_x - P_y\right)\cos\gamma\right]dA$$

Where the directional cosine version of the unit normal, $\hat{n} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle$ is used.

Proof: According to the hypotheses, the surface S is given by $z = f(x, y)$ where $f(x, y)$ is defined on some region S^* in which Green's theorem is true. Let $F^*(x,y) = \langle\eta(x,y), \zeta(x,y)\rangle$ be the vector field on S^* given by

$$\eta(x,y) = P(x,y,f(x,y)) + R(x,y,f(x,y))f_x(x,y)$$
$$\zeta(x,y) = Q(x,y,f(x,y)) + R(x,y,f(x,y))f_y(x,y)$$

Note that this means
$$\frac{\partial\eta}{\partial y} = P_y + P_z f_y + R_y f_x + R_z f_y f_x + Rf_{xy}$$
$$\frac{\partial\zeta}{\partial x} = Q_x + Q_z f_x + R_x f_y + R_z f_x f_y + Rf_{yx}$$

10. Since $\vec{F} = \left\langle \frac{y^2}{x^2}, -\frac{2y}{x}\right\rangle = \vec{\nabla}f$ where $f = \frac{-y^2}{x}$, the work done by the force field $\vec{F}$ in moving an object from $(1,1)$ to $(4,-2)$ is $f(4,-2) - f(1,1) = \frac{-(-2)^2}{4} - \frac{-(1)^2}{1} = 0$.

11. If $\vec{F} = \langle x^2 - y, x + y^2\rangle$ then the counterclockwise circulation of $\vec{F}$ around the triangle with vertices $(0,0),(1,0),(1,1)$ is, by Green's theorem, $\oint_c \vec{F}\cdot\overrightarrow{dr} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}dA = \iint_D 2dA = 1$..Joule, or some of energy. The outward flux is

$$\oint_c \vec{F}\cdot\vec{n}\,ds = \iint_D \nabla\cdot F\,dA = \iint_D 2x + 2y\,dA = \int_0^1\int_0^x 2x + 2y\,dy\,dx = \int_0^1 3x^2dx = 1.$$

12. a) A field is conservative if its curl is zero. Here,

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\cos y & -e^x\sin y & z \end{vmatrix} = 0 - 0 - e^x\sin y + e^x\sin y = 0$$, so yes, it's conservative.

b) For $f(x,y,z) = xy + xz + yz + 17$, $\overrightarrow{\nabla f} = \langle y+z, x+z, x+y\rangle$

13. The line segment joining $(0,0,0)$ and $(0,3,4)$ can be parameterized by $\vec{r}(t)=\langle 0,3t,4t\rangle$, $0\le t\le 1$.

Thus $\int_c x^2dx+yzdy+y^2dz=\int_0^1 36t^2+36t^2dt=24$.

14. The sphere can be parameterized by $\vec{r}(\theta,\phi)=\langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$ so that a normal vector is given by

$$\vec{r}_\phi\times\vec{r}_\theta=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi\\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0\end{vmatrix}=\langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\cos\phi\sin\phi\rangle.$$

Evaluating the vector field $\vec{F}(x,y,z)=\langle zx,xy,z^2\rangle$ on the surface of the sphere we have

$$\vec{F}(a\sin\phi\cos\theta,a\sin\phi\sin\theta,a\cos\phi)=\langle a^2\cos\phi\sin\phi\cos\theta, a^2\cos\phi\sin\phi\sin\theta, a^2\cos^2\phi\rangle$$

So the integrand for the flux integral is

$$\langle a^2\cos\phi\sin\phi\cos\theta, a^2\cos\phi\sin\phi\sin\theta, a^2\cos^2\phi\rangle\cdot\langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\cos\phi\sin\phi\rangle$$

$$=a^4\left(\cos\phi\sin^3\phi\cos^2\theta+\cos\phi\sin^3\phi\sin^2\theta+\sin\phi\cos^3\phi\right)=a^4\left(\cos\phi\sin^3\phi+\sin\phi\cos^3\phi\right).$$

Thus the flux is $a^4\int_0^{\pi/2}d\theta\int_0^{\pi/2}\left(\cos\phi\sin^3\phi+\sin\phi\cos^3\phi\right)d\phi=\frac{a^4\pi}{2}\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi=\frac{\pi a^4}{4}$

15. The divergence (Gauss') theorem says that the flux is

$$\iint_S \vec{F}\cdot\overrightarrow{dS}=\iiint_E\vec{\nabla}\cdot\vec{F}dV=\iiint_E 2x+1dV=\int_0^\pi\int_0^{2\pi}\int_0^2(2\rho\sin\phi\cos\theta+1)\rho^2\sin\phi d\rho d\theta d\phi$$

$$=\int_0^\pi\int_0^{2\pi}\int_0^2\left(2\rho^3\sin^2\phi\cos\theta+\rho^2\sin\phi\right)d\rho d\theta d\phi=\int_0^\pi\int_0^{2\pi}8\sin^2\phi\cos\theta+\frac{8}{3}\sin\phi d\theta d\phi=$$

$$=\frac{16\pi}{3}\int_0^\pi\sin\phi d\phi=\frac{32\pi}{3}.$$

16. By Stoke's theorem,

$$\iint_S\vec{\nabla}\times\vec{F}\cdot\hat{n}dS=\int_{\partial S}\vec{F}\cdot\overrightarrow{dr}=\int_0^{2\pi}\left\langle 18\cos^2t\sin t,\left(9\cos^2t+16\sin^4t\right)^{3/2},\sin(1)\right\rangle\cdot\langle -3\sin t,2\cos t,0\rangle dt$$

$$\int_0^{2\pi}\left(-54\cos^2t\sin^2t+2\left(9\left(1-\sin^2t\right)+16\sin^4t\right)^{3/2}\cos t\right)dt=\int_0^{2\pi}-6\sin^2 2tdt+2\int_0^0\left(16u^4-9u^2+9\right)du$$

$$=-\frac{27}{2}\int_0^{2\pi}\sin^2 2tdt=-\frac{27}{4}\int_0^{2\pi}\ 1-\cos 4t\,dt=-\frac{27\pi}{2}.$$